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Dynamical multiscaling in quenched Skyrme systems

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Abstract. – Strong dynamical scaling violations exist in quenched two-dimensional systems with vector O(3) order parameters. These systems support non-singular topologically stable configurations (skyrmions). By tuning the stability of isolated skyrmions to expand or shrink, we find dramatic differences in the dynamical multiscaling spectrum of decaying moments $\langle|\rho|^n\rangle$ of the topological charge density distribution and in particular in the decay of the energy density $\epsilon \sim \langle|\rho|\rangle$. We present a simple two–length-scale model for the observed exponents in the case when isolated skyrmions expand. No such simple model is found when isolated skyrmions shrink.

Phase-ordering systems quenched from disordered initial conditions into an ordered phase typically evolve correlations that dynamically scale—they are self-similar in time [1]. Dynamical scaling provides a powerful framework for the analysis of phase-ordering exponents and correlations. To understand scaling, systems that *violate* scaling are important to study. Logarithmic scaling violations exist in the spherical limit of conserved vector systems [2]. Recently, strong scaling violations have been found in one-dimensional [3] and two-dimensional (2D) [4], [5] systems supporting non-singular topological textures ("skyrmions").

In this paper we investigate the phase-ordering process of the 2D O(3) vector model in more detail, paying particular attention to sub-dominant terms in the free energy. These terms control the stability of individual skyrmions towards shrinking or growing in size. Surprisingly, these sub-dominant terms dramatically change the exponent ϕ_N characterizing the decay of energy in the system.

We restrict ourselves to dissipative noise-free "model-A" dynamics, $\partial_t\vec{m} = -\delta F/\delta\vec{m}$, where $|\vec{m}| = 1$ is maintained as a constraint. Our effective coarse-grained free energy is

$$F[\{\vec{m}\}] = \frac{1}{8\pi}\int \mathrm{d}^2\mathbf{r}\,(\nabla\vec{m})^2 + \pi\theta\int d^2\mathbf{r}\,\rho^2, \tag{1}$$

where the topological charge density ρ is

$$\rho(\mathbf{r}) = \vec{m} \cdot (\partial_x \vec{m} \times \partial_y \vec{m})/4\pi, \tag{2}$$

and is locally conserved in continuum systems [6]. The term $\pi\theta \int \mathrm{d}^2\mathbf{r}\, \rho^2$ in eq. (1) is analogous, when θ is positive, to the Skyrme term used to stabilize skyrmion configurations in high-energy models [7]. At time $t = 0$, we randomly orient the unit vector $\vec{m}$ at each site. This initial condition corresponds to a quench from $T = \infty$ to $T = 0$.

A 2D skyrmion (see, *e.g.*, [6]) is a field configuration $\vec{m}(\mathbf{r})$ that wraps around the O(3) order-parameter sphere exactly once. It is described by a spatially extended topological charge $\rho(\mathbf{r})$ that integrates to ± 1. Each skyrmion can be characterized by a size, defined, *e.g.*, as the half-width of the typically bell-shaped spatial distribution of topological charge $\rho(\mathbf{r})$. In a 2D O(3) phase-ordering system, which has a mixture of skyrmions and anti-skyrmions, we can define at least three natural length-scales: the average skyrmion size $L_T \sim t^{\phi_T}$, the average separation of skyrmions $L_N \sim t^{\phi_N}$, and a length characterizing positive and negative charge separation—for example the inverse interface density, $L_C \sim t^{\phi_C}$, of $\rho = 0$ contours. All three length-scales are found to have distinct growth exponents.

The static (minimal energy) configurations of an infinite *continuum* system with a pure quadratic gradient energy ($\theta = 0$) were studied by Belavin and Polyakov (BP) [8]. For any net number of skyrmions in the system, they found a class of minimal energy states that are degenerate with respect to both the scale and position of the individual skyrmions. Essentially, the skyrmions in the pure skyrmion (or pure anti-skyrmion) BP configurations do not interact, and they neither expand nor shrink.

In generic systems, the energy density will also contain higher-order gradient terms, arising, *e.g.*, from lattice interactions. The dominant corrections at late times (and hence large length-scales) in quenched systems will come from fourth-order gradient terms, such as the Skyrme term in eq. (1). Such terms will destabilize the individual skyrmions towards expanding or shrinking. By adding the Skyrme term in eq. (1) with a sufficiently large amplitude $\theta > \theta_{\mathrm{c}}$, it is possible to ensure that all skyrmions expand [9]. In our simulations, we use a 9-point Laplacian with nearest– and next-nearest–neighbor interactions and an isotropic fourth-order term. Our leading interactions in Fourier space are proportional to $(k^2 - k^4/24)|\vec{m}_{\mathbf{k}}|^2$, resulting in a positive θ_{c}.

Previous work [4], [5], [11], [12] has considered systems where isolated skyrmions shrink and ultimately unwind through the lattice ($\theta < \theta_{\mathrm{c}}$), thus violating charge conservation. For $\theta > \theta_{\mathrm{c}}$, topological charge density is eliminated solely through the mutual annihilation of regions of positive and negative ρ —a process conserving the topological charge. For $\theta < \theta_{\mathrm{c}}$, skyrmion unwinding and skyrmion/anti-skyrmion annihilation are competing processes —a partially annihilated skyrmion cannot unwind by itself [5]. We re-analyze our earlier results [5], and also study systems with $\theta > \theta_{\mathrm{c}}$ using the same numerical techniques. We use $\theta = 0.012$ in dimensionless units with a square-lattice spacing of 0.01. Our results are qualitatively unchanged for $\theta = 0.006$, with identical exponents. This indicates that the value $\theta = 0.012$ is well above θ_{c}. The results presented below were obtained in a system of size 512×512, and averaged over 10 (respectively, 12) runs in the case $\theta = 0$ (respectively, $\theta = 0.012$).

To characterize the system in the absence of dynamical scaling of correlations, we measure the asymptotic (late-time) decay exponents β_n of the moments of the topological charge density distribution $P(|\rho|, t)$:

$$\langle |\rho(\mathbf{r}, t)|^n \rangle \sim t^{-\beta_n}\,, \tag{3}$$

where $\langle \cdots \rangle$ denotes a spatial average. (We assume and observe power law decay with time.) The moment analysis in terms of β_n is analogous to multifractal analysis of static fractal [13]

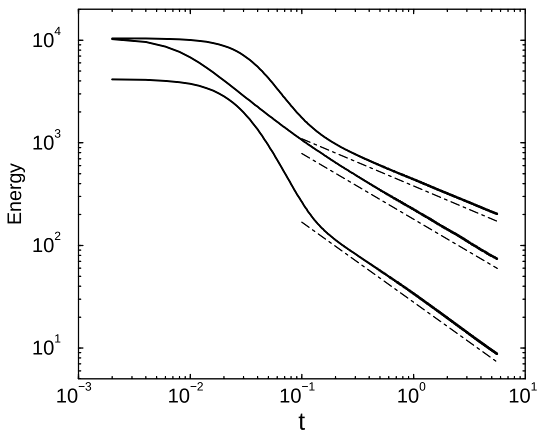

Fig. 1

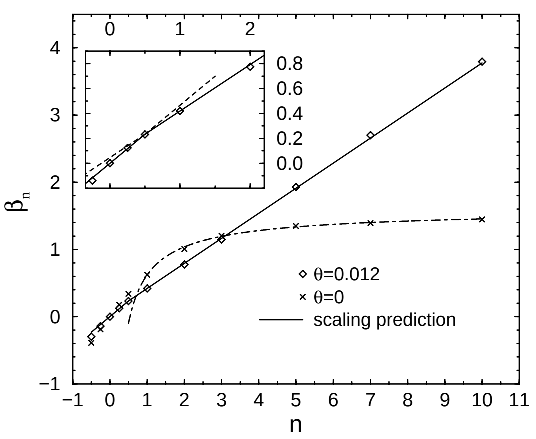

Fig. 2

Fig. 1. – The exchange energy (first term of eq. (1)) for Skyrme-term amplitudes $\theta = 0.012$ (upper curve) and $\theta = 0$ (middle curve). The lower curve shows the Skyrme energy (second term of eq. (1)) in the case $\theta = 0.012$. Even though the exchange energy dominates the total energy at late times, the decay exponent of the exchange energy depends on the Skyrme-term amplitude θ. The straight dot-dashed lines have slopes -0.46 (upper line), -0.64 (middle), and -0.78 (lower).

Fig. 2. – Growth exponents β_n for the n-th moment of the topological charge density, eq. (3). Shown are the β_n spectra for a system with Skyrme-term amplitude $\theta = 0 < \theta_c$ and with $\theta = 0.012 > \theta_c$. Error bars do not exceed the size of the symbols. The full line shows the two–length-scale prediction of eq. (4) with $\phi_T = 0.187$ and $\phi_N = 0.210$. The dot-dashed line shows the phenomenological fit to the β_n spectrum in the $\theta = 0$ case, given by eq. (5) with $\phi_N = 0.31$ and $p = 0.78$. Note that this fit is incompatible with the data for $n < 1$. The inset highlights more clearly the discontinuity at $n = 1/2$ in the slope of the β_n curve for $\theta = 0.012$, where the dashed lines are linear extrapolations of the regions above and below $n = 1/2$.

or turbulent [14] systems, only with the measuring length-scale replaced by the inverse time elapsed since the quench (see, *e.g.*, [15]).

The results for the asymptotic evolution of the energy density ϵ are dramatically different for $\theta < \theta_c$ and $\theta > \theta_c$. At late times, ϵ is dominated by the total number density of skyrmions and anti-skyrmions, so that $\epsilon \sim \langle|\rho|\rangle \sim 1/L_N^2 \sim t^{-2\phi_N}$ and $\beta_1 = 2\phi_N$. Whereas systems with skyrmions unstable towards shrinking ($\theta < \theta_c$) have $\phi_N = 0.32 \pm 0.01$, systems with $\theta > \theta_c$ have $\phi_N = 0.23 \pm 0.01$, as shown in fig. 1. (Here we extract ϕ_N from the exchange energy, which dominates the total energy at late times (see fig. 1). The values extracted directly from $\langle|\rho|\rangle$ are $\phi_N = 0.31 \pm 0.01$ and $\phi_N = 0.21 \pm 0.01$, respectively, and are consistent. Error bars shown are always statistical errors.) To our knowledge this is the first example of a phase-ordering system in which a sub-dominant interaction controls the growth laws.

Higher moments of the topological charge density, eq. (3), are shown in fig. 2. The β_n are multiscaling —they are not simply proportional to n. This indicates that $P(|\rho|, t)$, shown in fig. 3, is determined by more than one length-scale.

We can explain many of the observed features of $P(|\rho|, t)$ using the profile of a single Belavin-Polyakov (BP) skyrmion in combination with the multiple length-scales L_T, L_N, and L_C. We first discuss the case $\theta > \theta_c$, where (after initial transients due to large gradients) skyrmions grow and annihilate with anti-skyrmions, and there is no unwinding through the lattice.

For $\theta > \theta_c$, a scaling argument involving both L_T and L_N suffices to characterize all of the β_n.

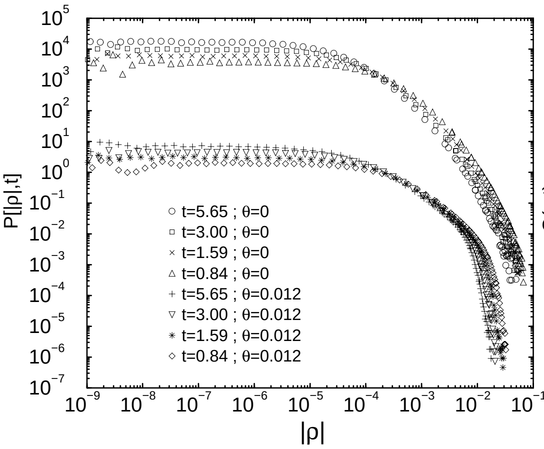

Fig. 3

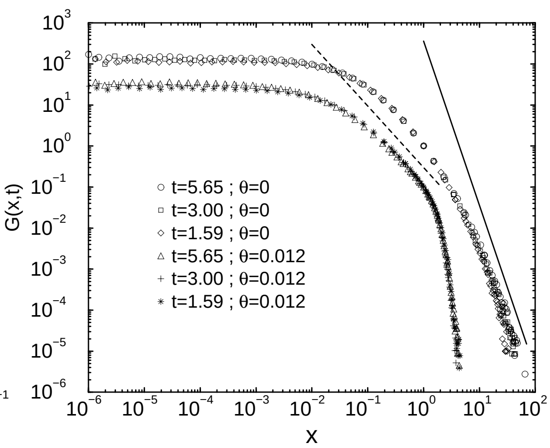

Fig. 4

Fig. 3. – The distribution $P(|\rho|, t)$ of topological charge density at indicated times t. The evolution of the small-ρ regime is well described by $P(0,t) \sim t^{0.75\pm0.05}$ in both cases. The vertical axis of the $\theta = 0.012$ curves is offset by three decades for clarity.

Fig. 4. – Re-scaled distributions $G(x, t)$ of topological charge density at indicated times t. The vertical axis of the $\theta = 0.012$ curves is offset by one decade for clarity. The horizontal and vertical axes of the original $P(|\rho|, t)$ curves have been rescaled so that all curves collapse at $x = 1$. The straight lines drawn in the figure have slopes $-3/2$ (dashed line) and -4 (solid line).

Picture the system divided into regions of size L_N, each containing a skyrmion or anti-skyrmion of size L_T (see figures in [4], [5]). Represent each skyrmion as an isolated BP configuration of scale L_T, so that $\rho(\mathbf{r}) = \pi^{-1}L_T^2/(L_T^2 + r^2)^2$ [6] and $\langle|\rho(\mathbf{r},t)|^n\rangle \propto {L_N}^{-2}L_T^{2n}\int_0^{L_N} r\,\mathrm{d}r\,(r^2 + L_T^2)^{-2n}$. For $n > 1/2$, the integral is dominated by the central region of radius $r \le L_T$, and we obtain $\langle|\rho(\mathbf{r},t)|^n\rangle \sim L_T^{2-2n}L_N^{-2}$. For $n < 1/2$, on the other hand, the integral is dominated by the large-r regions, giving $\langle|\rho(\mathbf{r},t)|^n\rangle \sim L_T^{2n}L_N^{-4n}$. For $n = 1/2$, $\langle|\rho(\mathbf{r},t)|^{1/2}\rangle \sim L_T L_N^{-2}\log(t)$. In addition, the mixture of skyrmions and anti-skyrmions in the quenched system requires the presence of contours on which $\rho = 0$ (probed by the third length, L_c, which is discussed later). We observe $P(|\rho|) = \text{const}$ for small $|\rho|$, which corresponds to a linear profile $\rho(z) \sim z$ with distance z away from the $\rho = 0$ contours. This constant regime for small ρ implies that moments with $n \le -1$ diverge, and the corresponding β_n are not defined. Summarizing:

$$\beta_n = \begin{cases} 2n\phi_T + 2(\phi_N - \phi_T), & n \ge 1/2, \\ n(4\phi_N - 2\phi_T), & -1 < n \le 1/2. \end{cases} \tag{4}$$

This piecewise-linear prediction agrees well with our data for $\theta > \theta_\mathrm{c}$ in fig. 2. The best fit is obtained for the values $\phi_N = 0.210 \pm 0.005$ and $\phi_T = 0.18 \pm 0.02$ [16].

For $\theta > \theta_\mathrm{c}$, we show our data for the distribution $P(|\rho|, t)$ in fig. 3 (lower set of curves). The $P(|\rho|) = \text{const}$ regime is visible for small $|\rho|$, followed by a developing power law regime for intermediate $|\rho|$ and an exponential cut-off at large $|\rho|$. In fig. 4, the data from fig. 3 are rescaled to test the scaling ansatz $P(|\rho|, t) = f_1(t)G[|\rho|f_2(t)]$, where f_1 and f_2 are functions of time only. A good quality of collapse is achieved for large and intermediate x, where $x = |\rho|f_2$. However, significant scaling violations are seen at small x.

The distinct growth laws for L_N and L_T are sufficient to introduce scaling violations in $P[|\rho|, t]$ [17]. The approximate power law behavior $G[x] \sim x^{-3/2}$ observed in the intermediate-x

range (see fig. 4) is consistent with a BP-like profile of the skyrmions. For a skyrmion of size L_T, $\rho(\mathbf{r}) = \pi^{-1}L_T^2/(L_T^2+r^2)^2$ implies the topological charge distribution of a single skyrmion $P_{L_T}(\rho) = \frac{\sqrt{\pi}}{2}L_T\rho^{-3/2}$ for $\rho \leq \rho_{\max} = \pi^{-1}L_T^{-2}$. However, the tail of the BP profile is expected to be cut off at $r \geq L_N$ due to the presence of neighboring skyrmions and anti-skyrmions; this corresponds to ρ values $\rho \geq \rho_c \sim t^{2\phi_T - 4\phi_N}$. Since $\rho_{\max}/\rho_c \sim t^{4(\phi_N-\phi_T)}$, the width of the $G[x] \sim x^{-3/2}$ regime is predicted to increase with t—hence violating the scaling ansatz. Our data is consistent with this, and the lack of scaling of $P(|\rho|,t)$ is directly confirmed by the observed multiscaling character of β_n.

For $\theta < \theta_c$, the simple picture presented above does not apply. The instability of skyrmions towards unwinding in this case leads to the presence of small skyrmions that significantly change $P(|\rho|,t)$ and hence β_n. Furthermore, we show below that unwinding skyrmions *cannot* be treated independently of annihilation. To proceed, we consider the distribution density $H(R,t)$ of skyrmion sizes R at time t [18].

We see in fig. 2 that β_n saturate asymptotically at large n, for $\theta = 0$. This indicates the presence of small unwinding skyrmions, which dominate the moments of $P(|\rho|,t)$ for large enough n. Let us first assume that small skyrmions (with sizes $R \ll L_T$) do not participate in annihilation processes and unwind independently. In that case, the size distribution function $H(R,t)$ satisfies a continuity equation $\partial_t H + \partial_R J = 0$, where the "size current" is (see ref. [9]) $J = H\partial_t R \sim -H/R^3$. To maintain $\dot{N}$, where N is the total number density of skyrmions and antiskyrmions, we must have a constant flux $J(R) = J(\xi) \sim \dot{N}$ (where ξ is the lattice scale), or $H \sim -\dot{N}R^3$ for small R. These small skyrmions would dominate the high β_n, so that $\langle|\rho|^n\rangle \sim \int_\xi^{L_N} \mathrm{d}R\, H(R)R^{2-2n} \sim \dot{N}\xi^{6-2n} \sim t^{-(1+2\phi_N)}$ for $n \geq 3$.

For $\theta = 0$, the measured β_n do approach $1 + 2\phi_N$. This indicates that unwinding is a significant relaxation process at late times. However, they are not constant for large n but instead fit the empirical form

$$\beta_n = 1 + 2\phi_N - n^{-p} \tag{5}$$

within error bars for all $n \geq 1$ (the unit prefactor of the power law term ensures that $\beta_1 = 2\phi_N$). In fig. 2, we plot the best fit, with $\phi_N = 0.31$ and $p = 0.78$ ($\phi_N = 1/3$ and $p = 2/3$ provide a comparable fit). Our data for $P(|\rho|,t)$ are shown in fig. 3 for $\theta = 0$ (upper set of curves). Similarly to the $\theta > \theta_c$ case, we see a $P(|\rho|) = \text{const}$ regime at low $|\rho|$. As seen in fig. 4, $P(|\rho|,t)$ does not scale well, even for large x. An asymptotic power law tail $P(|\rho|,t) \sim |\rho|^{-4}$ would lead to β_n saturating for $n \geq 3$. The asymptotic approach *towards* that power law (shown in fig. 3) is another manifestation of β_n asymptotically *approaching* $1 + 2\phi_N$ at large n. From the previous paragraph, the existence of the non-zero correction term n^{-p} for $n \geq 3$ is *inconsistent* with having a sub-population of independently unwinding BP skyrmions.

Two equivalent mechanisms can describe this breakdown of the picture of small unwinding BP skyrmions for $\theta < \theta_c$. Either the profile of individual shrinking skyrmions is asymmetric and hence not BP-like, or the partial annihilation of skyrmions with anti-skyrmions changes the unwinding process [19]. In comparison, high-order gradient terms on their own do not significantly affect the structure of an unwinding skyrmion until it approaches the lattice scale.

On the other hand, does unwinding affect the structure and evolution of annihilating skyrmions and anti-skyrmions? To address this question, we need to probe the *large*-scale spatial distribution of the skyrmions and anti-skyrmions. We can do this by measuring the interface line density of $\rho = 0$. We measure this density, $L_C^{-1} \sim t^{-\phi_C}$, by counting the nearest-neighbor lattice sites between which ρ changes sign, and find $\phi_C = 0.41 \pm 0.01$ for both $\theta < \theta_c$ and $\theta > \theta_c$. This exponent is significantly greater than ϕ_T or ϕ_N, and is another demonstration of the strong scaling violation seen in our system.

In addition to L_C, we have measured two other quantities that probe the large-scale structure of the system. The low-ρ value of the topological charge density $P(0,t) \sim t^{0.75\pm0.05}$ probes the charge distribution near the $\rho = 0$ contours, while the half-width $L_\rho \sim t^{0.22\pm0.03}$ of ρ-ρ correlations provides a measure of the typical skyrmion size. The agreement of the given values of the exponents of L_C, $P(0,t)$, and L_ρ in the $\theta > \theta_c$ and $\theta < \theta_c$ cases is striking —especially considering the dramatic differences in the β_n spectrum.

We conclude simply: annihilation affects unwinding, but not vice versa.

In summary, we have studied phase-ordering dynamics in 2D O(3) systems supporting skyrmions. By tuning the strength θ of the Skyrme term, we made isolated skyrmions unstable to shrink and unwind ($\theta < \theta_c$) or to expand ($\theta > \theta_c$). We found dramatic changes in the multiscaling spectrum β_n of moments of the topological charge density between the two cases. Having expanding skyrmions leads to a piecewise linear β_n spectrum explained by taking into account that the average skyrmion size $L_T(t)$ and the average skyrmion separation $L_N(t)$ scale differently in time. Having shrinking skyrmions leads to a curved β_n spectrum remarkably well fit by eq. (5).

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[9] Our Skyrme term in eq. (1) does not describe all possible fourth-order gradient terms. For general fourth-order terms, the energy of a spherically symmetrical skyrmion of size R is $E(t) \sim 1 + A(\theta - \theta_c)R^{-2}$, where A is a positive constant. The rate of change of the energy is $\partial_t E \sim \dot{R}^2$ [10]. Self-consistency gives $\partial_t R \sim -R^{-3}$, which leads (in our case) to $R(t) \propto [(\theta - \theta_c)(t - t_0)]^{1/4}$. Skyrmions are also destabilized towards shrinking, with a similar rate, in the *continuum* model when a soft-spin potential that allows variations in the spin magnitude is used [4].

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[16] These exponents are distinct, as can be seen from the distinct linear regimes shown in the inset of fig. 2. In light of this, there is covariance in the measured errors.

[17] Additional scaling violations arise at small x arise due to a third length-scale L_C.

[18] To obtain eq. (4), we effectively used $H(R,t) = \delta\left[R - L_T(t)\right]/L_N^2$.

[19] In the continuum limit, any non-BP structure in a skyrmion can be described by an additional anti-skyrmion component [8].